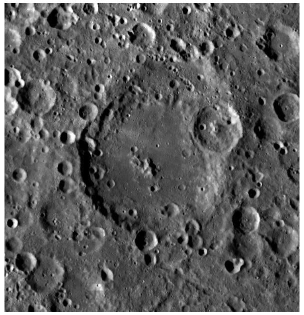
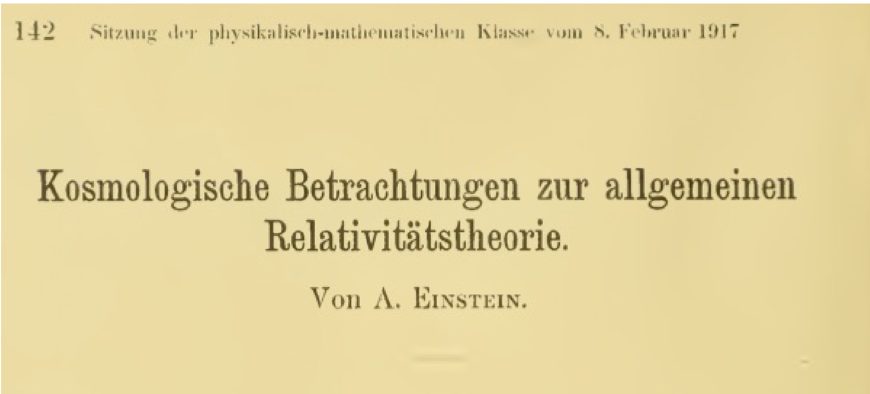


The Mach–Einstein Principle of 1917–1918

Gordon Belot
University of Michigan



142 Sitzung der physikalisch-mathematischen Klasse vom 8. Februar 1917

Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie.

Von A. Einstein.

In his cosmology paper of 1917, Einstein argued in favour of the finitude of space, presented the first relativistic cosmological model (his eternal and time-independent universe), and concluded that the equations of general relativity had to be altered.

The Argument for Spatial Finitude

1) "we must supplement the differential equations by limiting conditions at spatial infinity, if we really have to regard the universe as being of infinite spatial extent."
2) The most obvious supplement—imposition of asymptotic flatness at spatial infinity—has undesirable consequences:
    a) violation of general covariance;
    b) violation of Mach's Principle;
    c) no equilibrium for island universes.
3) It appears that there are no other acceptable boundary conditions.

So we should abandon spatial infinitude.

Two Views of General Relativity

* Misner, Thorne, and Wheeler:
  "Space tells matter how to move
  Matter tells space how to curve"†
* Einstein:
  "It seems to me absurd to ascribe physical properties to 'space.' The totality of masses produces the $g_{\mu\nu}$-field (gravitational field), which in turn governs the course of all processes, including the propagation of light rays and the behavior of measuring rods and clocks."‡

† *Gravitation*, 5.
‡ Letter to Mach of late December 1913 (5.495). Paraphrase: Matter tells matter how to move.

The Mach–Einstein Principle

"The G-field is completely [restlos] determined [bestimmt] by the masses of the bodies. Since mass and energy are—according to the special theory of relativity—the same, and since energy is formally described by the symmetric energy tensor ($T_{\mu\nu}$), it follows that the G-field is conditioned and determined [bedingt und bestimmt] by the energy tensor of matter."

From the reply to Kretschmann of March 1918—On the Foundations of the General Theory of Relativity (7.4). Translation of CPAE emended.

See also: the letter to Besso of sometime after 9 March 1917 (8.308); the letter to de Sitter of 24 March 1917 (8.317); the letters to Mie of 22 December 1917 (8.416), 8 February 1918 (8.460), and 22 February 1918 (8.470); and the letter to Sommerfeld of 1 February 1918 (8.453).

* Not much like anything Mach ever said.
* There were other Einstein's Mach's Principles before 1916 and after 1918.
* Nothing to do with rotation.
  - If (M, g) is a homogenous solution of the Einstein-dust equations with vanishing shear and expansion, then (M, g) is either the Einstein static universe (if $\Lambda > 0$) or Gödel spacetime (if $\Lambda < 0$).†
  - So, plausibly, one can characterize the Gödel geometry in terms of material degrees of freedom, so it is not a counter-example to the present form of Einstein's Mach's Principle.

† Ozsváth, New Homogeneous Solutions of Einstein's Field Equations with Incoherent Matter Obtained by a Spinor Technique, §9.

Weakness

Einstein is clearly committed to at least the following:

> The Weak Mach–Einstein Principle: spacetime geometry supervenes on the disposition of matter (over time).

In other dialects:

* fixing the history of the distribution of matter fixes the spacetime geometry;
* there is only one spacetime geometry consistent with any given history of matter;
* the degrees of freedom of the matter-geometry system are exhausted by the material degrees of freedom.

Einstein Against Weakness (I)

* One way to specify the disposition of matter is to say: there is none, ever.
* It is consistent with the weak Principle that there be a groundstate of the gravitational field that obtains in this case.
* Einstein intends something stronger: in his reply to Kretschmann (7.4) of 1918 he asserts that his original field equations violate Mach's Principle because they countenance Minkowski spacetime as a vacuum solution.

– Note that in this era Hilbert, too, thought that Minkowski spacetime was the only non-singular vacuum solution—see his 1916/17 lecture notes, §50 and Foundations of Physics (Second Communication).

– As late as 1932 the existence of further nonsingular vacuum solutions was considered an open question—see Lanczos, On the Problem of Regular Solutions of Einstein's Gravitational Equations.

– Choquet-Bruhat reports that in the early 1950s Einstein still maintained that Minkowski spacetime was the only non-singular vacuum solution asymptotically flat at spatial infinity—*A Lady Mathematician in this Strange Universe*, 118.

And in a paper submitted one day later:

> "If the De Sitter solution were valid everywhere, it would show that the introduction of the "λ-term" did not fulfill the purpose that I intended. Because, in my opinion, the general theory of relativity is a satisfying theory only if it shows that the physical qualities of space are completely determined [vollständig bestimmt] by matter alone. Therefore, no $g_{\mu\nu}$-field must exist (that is, no space-time continuum is possible) without the matter that generates it [welche es erzeugt]."

Critical Comment on a Solution of the Gravitational Field Equations Given by Mr. De Sitter (7.5) (1918). See also the letter to de Sitter of 24 March 1917 (8.317).

"It can be put jokingly this way. If I allow all things to vanish from the world, then, following Newton, the Galilean inertial space remains; following my interpretation, nothing remains."

Letter to Schwarzschild of 9 January 1916 (8.181). See also the letters to de Sitter of 24 March 1917 (8.317) and to Mie of 8 February 1918 (8.460).

EINSTEIN SEES END OF TIME AND SPACE

Destruction of Material Universe Would Be Followed by Nothing, Says Creator of Relativity.

Failing *New York Times,* 4 April 1921.

Suggestion: Einstein's Mach's Principle was something like the following:

> Facts about spacetime geometry (including, e.g., facts about the trajectories of freely falling bodies) supervene on facts about the distribution and motions of material bodies because the former sort of facts are wholly explanatorily dependent on the latter sort of facts.

Claim: For Einstein, the relevant sense of explanatory dependence was not causal.[†]

On this sort of view, geometric facts are genuine facts, but are ontologically derivative upon more fundamental facts about matter.

[†] So he is a relationalist in the sense of Dagupta, North, and Schaffer. And Belot (?).

In Support of this Suggestion

The 1913 letter to Mach quoted above. Plus a couple of texts of 1920:

* "according to the gen. theo of r., physical space has reality, but not an independent one, in that its properties are fully determined [vollständig bestimmt] by matter."[†]
* In relativity, space/ether "is by no means homogeneous, and its state has no independent existence, but rather depends [hängt ab] upon the field-generating [feld-erzeugenden] matter. Since the metric facts can no longer be separated in the new theory from the physical facts "proper," the concepts of "space" and "ether" flow into each other. Since the properties of space appear to be conditioned [bedingt] by matter, space is no longer a precondition [Vorbedingung] for matter in the new theory. The theory of space (geometry) can no longer be treated before or developed independently of mechanics and gravitation."[‡]

[†] Letter to Schlick of 30 June 1920 (10.67.)

[‡] Fundamental Ideas and Methods of the Theory of Relativity, Presented in Their Development (7.31), §22.

Does the (weak) Mach–Einstein Principle hold in general relativity?

Methodological stance: Interesting universal generalizations about general relativity are typically false; but it is often profitable to ask just what sort of counter-examples there are.

Background

Weakness & Strength

The Mach–Einstein Principle in General Relativity
Warmup: Maxwell's Theory
The Vacuum Case
Some Positive Results
Einstein vs. Einstein
Counter-Examples

In Conclusion

The Electromagnetic Mach–Einstein Principle

E-M M-E Principle: The disposition (over time) of (discrete) charged matter determines the electromagnetic field.

Warning: Einstein was no fan of the E-M M-E .[†]

[†] Einstein entertained/favoured an approach on which charged particles are stable structures in a non-linear electromagnetic field. See his letter to Lorentz of 23 May 1909 (5.163); On the General Theory of Relativity (Addendum) of November 1915 (6.22); The Foundations of the General Theory of Relativity of 1916 (6.30, 188); the letter to Kaluza of 29 May 1919 (9.48); his tribute to Lorentz of 1920 (Ether and Relativity, 7.38, 23).

An Approach with Honest Interactions

- We can get a well-behaved theory that includes self-interaction by following Abraham: work with extended particles that are rigid spheres in a preferred frame.
- Supervenience fails radically in the vacuum case: an infinite-dimensional family of solutions consistent with saying there is no charged matter anywhere, ever.
- But (plausibly): the only field consistent with a single charge permanently at rest is the charge's Coulomb field F_C.
  - (i) If the exterior field isn't spherically symmetric, then presumably the charge will accelerate at some point.
  - (ii) If the exterior field is spherically symmetric, if must be F_C (electromagnetic Birkhoff's theorem).
- If this works, then we have managed to prove that the EM-ME Principle holds in one simple case. If it doesn't work, it will be because there is some surprising way of evading (i)—and we can handle that by resorting to a little theft by adding a postulate that fields must share the symmetries of their sources.

One way to specify the history of massive matter is to say that there is none anywhere, ever. This is consistent with many different spacetime geometries.

So the Mach–Einstein Principle fails, unless we are willing to say that all of these geometries are unphysical (or that all but one are).[†]

[†] Doing so would not be especially costly: since there is massive matter in our world, whenever some process can be modelled using a vacuum solution, it had better be the case that it can also be modelled using a solution with matter.

Background

Weakness & Strength

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In Conclusion

The first substantive test of the Mach–Einstein Principle comes in the one-body case. The natural place to start is with a static isolated $\Lambda = 0$ body.

Fluid Stars

* Consider a static isolated blob of perfect fluid with constant mass density ρ and total mass M.
* If the exterior metric is static and asymptotically flat at spatial infinity, then:
  - the blob is a sphere (Lindblom, Masood-ul-Alam);
  - so the exterior metric is Schwarzschild (Birkhoff).

Elastic Bodies

* Set $G = 0$. Specify a compact elastic body $\mathcal{B}_0$ in equilibrium in Euclidean space.
* Then for each sufficiently small ε, there is a unique static spacetime that is asymptotically flat at spatial infinity and vacuum exterior to an elastic body $\mathcal{B}_\varepsilon$ that represents the deformation of $\mathcal{B}_0$ when $G = \varepsilon$.

Andersson *et al.*, Static Self-Gravitating Elastic Bodies in Einstein Gravity.

Their basic strategy: working in material variables, formulate the Einstein-elastic equations as a non-linear map between Sobolev spaces; take the derivative of this map at the relaxed configuration; project out symmetries and use the implicit function theorem to define $\mathcal{B}_\varepsilon$.

Moral

* So if we are able to prove (or willing to stipulate) that the geometry exterior to an isolated body must be asymptotically flat and have the same symmetries as the body, then the Mach–Einstein Principle holds for certain sorts of isolated bodies.
* Let us set aside for today the question whether the field equations imply that the geometry exterior to a static isolated body must be asymptotically flat (they don't).
* Plausibly, the geometry exterior to a spherically symmetric body should be spherically symmetric—otherwise, won't the symmetry of the body fail when it eventually feels the exterior asymmetry?
* Plausibly, for the same reason, the geometry exterior to a static body should be static.

Sciama

"I had the privilege of discussing this question with Einstein only a week before he died, in April 1955. It was at the end of my year's visit to the Institute for Advanced Study. To help ease my tension I started the discussion with a prepared sentence, 'Professor Einstein, I would like to talk about Mach's principle, and I have come to defend your former self against your later self.' Fortunately he laughed uproariously at this rather feeble beginning, perhaps to put me at my ease, and said, "That is gut, ja." Our subsequent discussion was rather inconclusive and soon wandered onto other topics. . ."

Issues in Cosmology, p. 396

Einstein to Pirani

* Mach "sought to abolish space and replace it by the relative mutual inertia of ponderable bodies. . . . This certainly did not work . . . ."
* When people "speak today about Mach's Principle they do not mean to abolish the continuum but to preserve the field. But they think that the field ought to be completely determined [völlig bestimmt] by matter. This however is a ticklish affair [eine heikle Sache] for the T_{ik} which are to represent 'matter' always presuppose the g_{ik}."
* "In my opinion we ought not to speak about the Machian Principle any more. It proceeds from the time in which one thought that the 'ponderable bodies' were the only physical reality and that all elements that could not be fully determined by them ought to be avoided in the theory. I am well aware that for a long time I too was influenced by this fixed idea."

Letter to Pirani of 2 February 1954. Translation of Torretti, Relativity and Geometry, 202.

Ehlers's Version of the Worry

* "If you have a [stress-energy] tensor $T_{\mu\nu}$ and not a metric, then this does not meaningfully describe matter. There is no theory of physics so far, which can describe matter without already the metric as an ingredient of the description of matter. Therefore within existing theories the statement that the matter by itself determines the metric is neither wrong nor false, but it is meaningless."

From a transcript of a conference—p. 93 in *Mach's Principle: From Newton's Bucket to Quantum Gravity.*
Mistakenly attributed to Einstein in Schmid, Cosmological Gravitomagnetism.

How Does it Work?

i) Matter enters the field equations only through the stress-energy tensor.
ii) So to say that matter determines the spacetime geometry is to say that there is only one metric consistent with a given stress-energy tensor.
iii) But the stress-energy tensor depends on the spacetime metric as well as the variables describing the matter content of spacetime.
iv) So in general relativity it makes no sense to speak of some way that matter might be distributed as determining (or failing to determine) the geometry of spacetime.

Rejoinders:

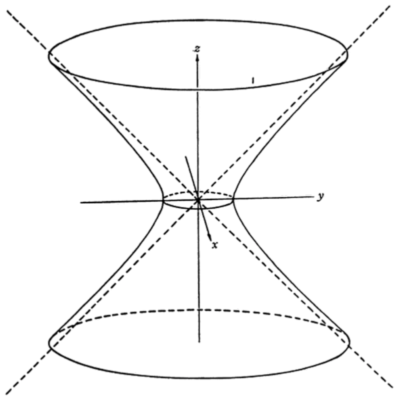
* The argument is not sound—it makes sense (because it is false) to say that the null distribution of matter determines the spacetime metric.
* More generally, we have just seen some tricks for specifying the matter distribution that do not involve specifying the stress-energy tensor and which do not seem to require that a spacetime metric have been specified in advance.
* Indeed, I can say: there is a blob of incompressible fluid of total mass M and that density ρ without having first specified a spacetime metric—since this claim makes sense in either Newtonian physics or general relativity.
* (But wait! Do we really have the same properties in both theories? I don't know. But we certainly have properties playing similar functional/operational roles—and that suffices here.)

A Fall-Back

The worst case would be: we cannot specify the state of matter without also specifying the geometry of the part of spacetime occupied by matter. Even in this case, all is not lost for the Mach–Einstein Principle: it is a substantive question whether there is ever more than one (inextendible) vacuum exterior solution consistent with a given way of specifying the material and geometric degrees of freedom in an occupied region.

That Pesky Constant

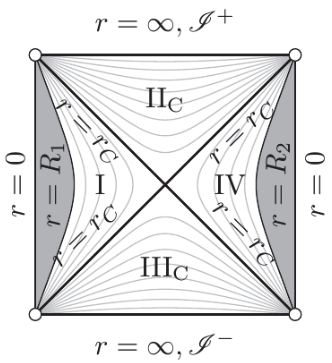
In order to construct his eternal, time-independent, spatially-finite cosmology, Einstein added a term to his equations that gives spacetime a repulsive tendency. Unchecked, it leads to an exponential spatial expansion.



de Sitter Spacetime

Contrary to Einstein's intentions, that makes it much easier to find counter-examples to the Mach–Einstein Principle: in the $\Lambda > 0$ regime, an isolated, static, spherically symmetric perfect fluid star admits more than one inextendible exterior geometry.

A $\Lambda > 0$ Universe Containing Two Stars

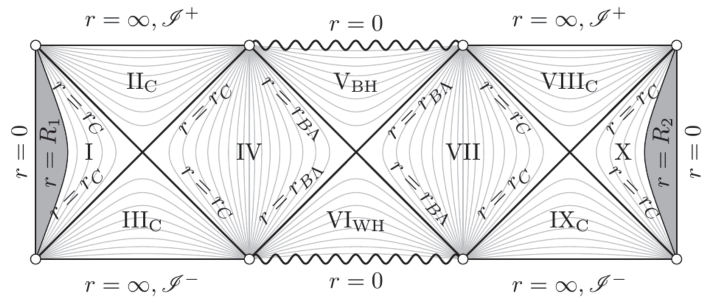


* Cauchy surfaces have topology S^3.
* The exterior spacetime is Schwarzschild–de Sitter.
* Regions I and IV are static.
* Regions II and III undergo exponential expansion/contraction.[†]
* We could quotient by a reflection symmetry to get a solution with a single star (with elliptic spatial topology).

Figure 2 of Andréasson *et al*,
Static Solutions of the Einstein–Vlasov System with a Nonvanishing Cosmological Constant.

[†] So the $\Lambda > 0$ field equations do not imply that the geometry exterior to a static body must be static (even in the spherically-symmetric case).

Another solution with the same topology and matter configuration—but now with a black hole and a white hole.

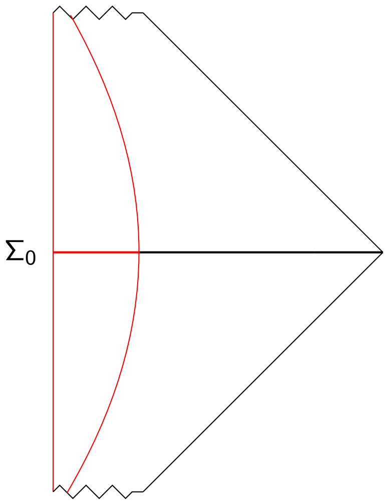


* For each $N = 1, 2, 3 \ldots$ there is an exterior of this kind with N back holes and N white holes interpolating between the two stars.
* We can again quotient to construct single-star versions.

Figure 3 of Andréasson *et al.*.
For more on quotients of such spacetimes, see Schleich & Witt, What Does Birkhoff's Theorem Really Tell Us?

Collapse

We can also construct counter-examples involving a single non-static but spherically symmetric $\Lambda = 0$ body.

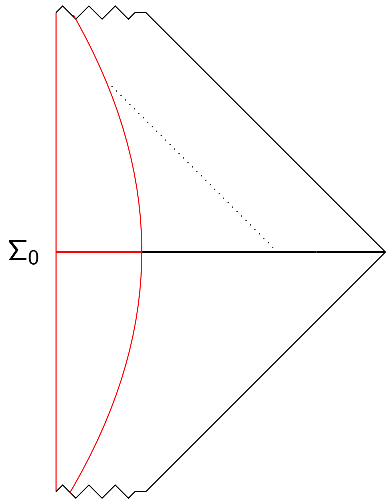


Our solution. The inner region of Σ_0 ($t = 0$) where the body is located is red. The outer region vacuum region of Σ_0 black.

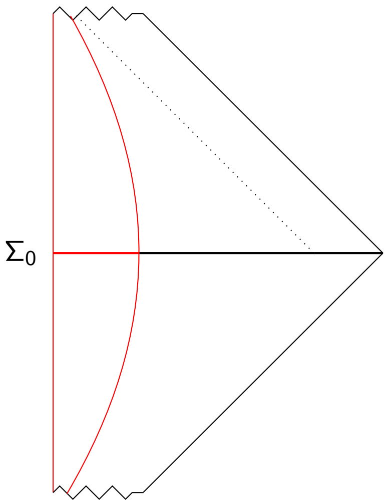
The simplest collapsing system:

* A collapsing ball of fluid surrounded by vacuum.
* We pose time-reversal-symmetric initial data at $t = 0$:
  - in the central region, a homogeneous sphere of fluid;
  - in the exterior region, Schwarzschild geometry.
* For positive t, the ball collapses under its own gravity, disappears behind a horizon, and a black hole forms leaving a future-eternal vacuum exterior.
* So for negative t, a past-eternal vacuum with a white hole—which is destroyed when an expanding ball of fluid emerges.

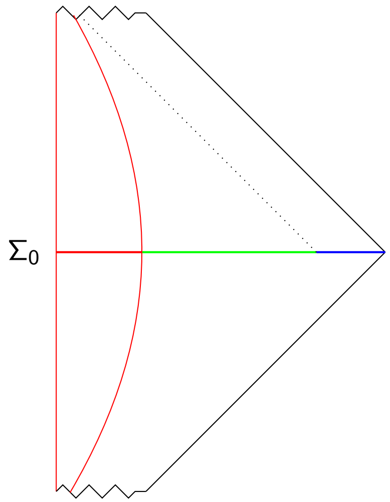
We are going to cook up another solution with different vacuum geometry near spatial infinity but with the same matter and geometry in the central region.



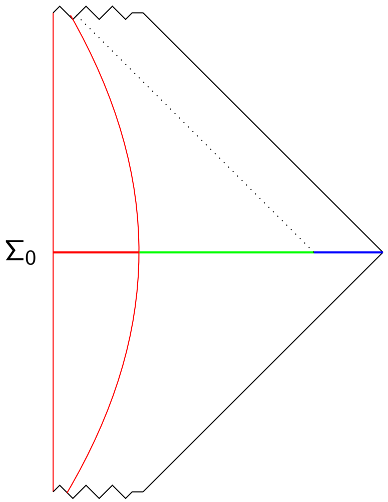
Some points in the exterior region of Σ_0 can send signals that will be received by the fluid body before it is destroyed.



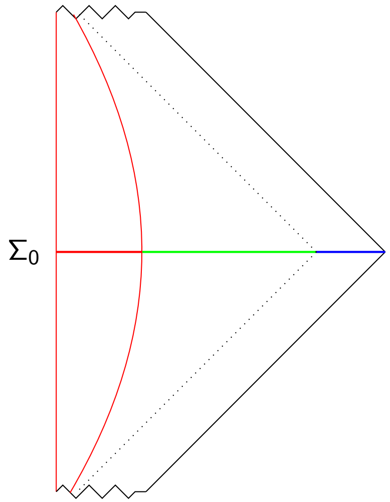
Signals sent inwards from points far enough out along Σ_0 will instead be snuffed out by the singularity left behind by the body.



Divide the exterior region Σ_0 into green (points that can signal the body) and blue (points that cannot).



Goal: Create a new initial data set on Σ_0 that matches the given one in the red and green regions but differs the blue region.



This will determine a spacetime with the same matter configuration as before—but with different geom near spatial/null infinity.

It is not *obvious* that we can do this:

* We cannot, for instance, just say that outside a transition zone, the blue region looks like a slice of Minkowski spacetime (because of the positive energy theorem).

But we *can* do it:

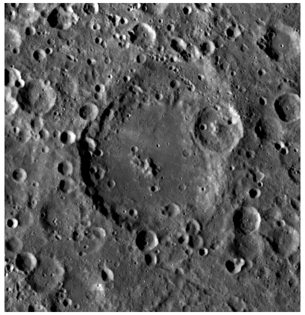
* A gluing technique of Chruściel and Delay allows us to pose asymptotically flat initial data on the blue region whose development is non-static.[†]
* Since the new solution is not static it cannot be spherically symmetric (Birkhoff).
* So here we have a vacuum exterior for a $\Lambda = 0$ spherically symmetric isolated body this is not spherically symmetric—and hence certainly differs from the spherically symmetric solution we started with.

[†] On Mapping Properties of the General Relativistic Constraints Operator in Weighted Function Spaces, with Applications.

Questions

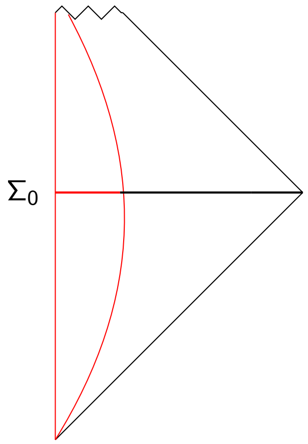
* What does Einstein's Mach's Principle require beyond mere supervenience?
* Is the Electromagnetic version of Einstein's Mach's Principle a consequence of the Maxwell–Abraham theory?
* In general relativity: what are the acceptable ways of specifying the material degrees of freedom upon which geometric degrees of freedom are supposed to supervene?
* How fares the Mach–Einstein Principle beyond the regime of static isolated bodies? What global structures are (in)consistent with the Principle?
* Can a static body admit exteriors with distinct asymptotic behaviours at spatial infinity?

Thank You!

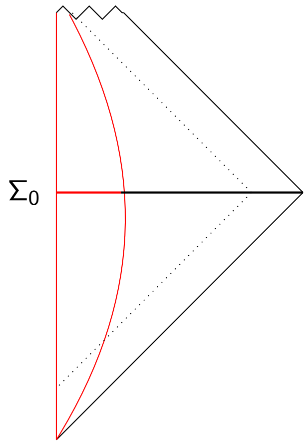


Mach

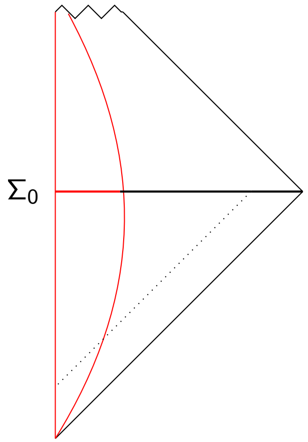
Einstein



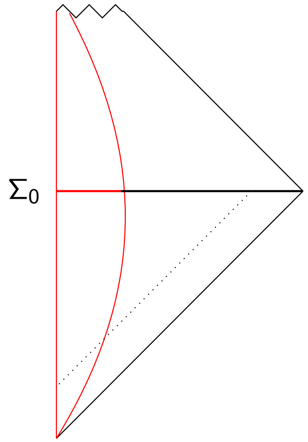
A spherically-symmetric collapsing body that exists eternally towards the past.



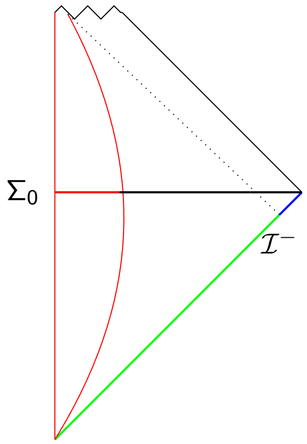
In this setting, even points on Σ_0 that cannot signal the body can be signalled by it.



So changing initial data in a region on Σ_0, no matter how far out, can be expected to require changes to the history of the body.



Possible way out: secure a notion of inward moving disturbance for initial data on Σ_0. (Far from obvious how to do this.)



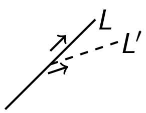
Possible way out: Pose initial data on $\mathcal{I}^-$ instead of on Σ_0.
Problem: extant results give existence only locally in time.

Einstein Against Weakness (II)

* Einstein found an approximate solution describing the exterior field of an isolated spherical massive body, and Schwarzschild the corresponding exact solution
* So far so good—one might think that they had started down the road to showing that an isolated spherical body determines a certain spatiotemporal geometry
* But that is not Einstein's view. In his treatment and in Schwarzschild's, the spacetime geometry is asymptotically flat at spatial infinity—as you proceed outwards along a spatial geodesic from the central body, the geometry rapidly approaches that of Minkowski spacetime. To Einstein's mind, this means that the Principle is violated because the central mass is not responsible for the geometry

Let L be the path of followed by a free particle and L' any curve that initially coincides with L, then diverges from it. Then:

> *"the relativistic point of view requires that the actually described path L be preferred over the, from the logical point of view, equally possible path L', on the basis of a real cause [Realursache], which has the preference of L over L' as a consequence. . . . Mathematically, this means: the $g_{\mu\nu}$'s must be determined [bestimmt] completely by the $T_{\mu\nu}$'s . . . .*
> *"This requirement is not satisfied by Newton's theory, but also just as little by mine as long as the world is conceived as quasi-Euclidean. For then the $g_{\mu\nu}$'s are predominantly fixed by nonrelativistic boundary conditions at infinity. Then no real cause exists for the preference of path L over certain other L''s . . . "*



Letter to Mie of of 22 February 1918 (8.470). For related discussions, see Cosmological Considerations in the General Theory of Relativity (6.43 §2), the letter to Besso of sometime after 9 March 1917 (8.308), and the letters to Mie of 8 February 1918 (8.460) and 22 December 1917 (8.416)

The reasoning appears to be:

* Suppose that all solutions of the field equations that represent isolated systems are asymptotically flat at spatial infinity
* Then Minkowski spacetime is in some sense the default spacetime geometry
  - it represents how the world would be in the absence of matter;
  - departures from flatness near gravitating bodies are due to those bodies
* But in regions of spacetime distant from any gravitating bodies, the spatiotemporal geometry will depart from that of Minkowski spacetime only to a minuscule extent
* So we have to admit that in such regions, the geometry is largely determined by the fact that Minkowski spacetime is playing the role of a default in this theory

So:

Einstein has in mind something stronger than the Weak Mach–Einstein Principle

The Causal Mach–Einstein Principle?

* In the letter to Mie quoted above, Einstein requires that there be a real cause [Realursache] for motions and seems to say that the Mach–Einstein Principle is the mathematical implementation of this requirement
* This suggests that intends a causal reading of the principle:

  Spacetime geometry depends causally on the distribution of matter (and on nothing else)

  Call this the Causal Mach–Einstein Principle

Against This Suggestion. I

* Einstein tends to say that matter is geometry is bestimmt (determined) or bedingt (conditioned/constrained) by the distribution of matter
* If this was intended in a causal sense, you might expect him to say that geometry is kausal bestimmt or kausal bedingt by the distribution of matter
* This happens once. In a paper of 1914, a character who has "natural intelligence but has learned neither geometry nor mechanics" suggests that the motions of bodies are at least partially causally determined by the fixed stars†
* But he never returns to this phrasing, except in passages deleted or revised in a manuscript draft of 1920‡

† The Relativity Problem (4.31), §II

‡ The manuscript draft of Ether and Relativity (7.38) features a use of kausal bestimmt in a paragraph later deleted. It also features a kausal bedingt formulation of the Mach–Einstein Principle—but kausal bedingt is revised to bestimmt. The phrase kausal bestimmt also occurs in the letter to Schottky of 10 October 1917 (8.388) and in the irritated letter to Wulf of 25 February 1921—but those uses are unrelated to the Mach–Einstein Principle

Against this Suggestion. II

About Realursache

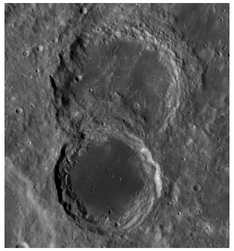
* Other than in the letter to Mie, not used formulations of the Mach–Einstein Principle
* A peculiar term that Einstein uses only four times, each time in implicit or explicit contrast with the merely factitious cause available in Newtonian and special relativistic physics to explain the asymmetry of Einstein's globes. But he never says that in general relativity matter is the Realursache of geometry†
* We see a similar pattern—using more causal language to complain about alternatives, less causal language to describe the situation in his theories—in "On the Present State of the Problem of Gravitation" of 1913 (4.17)

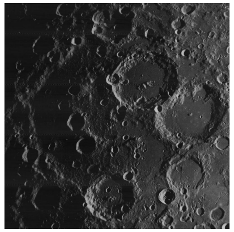
† The other three:
"Dialogue about Objections to the Theory of Relativity" of 1918 (7.13)
"Reply to Ernst Reichenbächer" of 1920 (7.49)
Nobel lecture of 1923 (14.75)

- A complaint about Nordström’s theory: “according to this theory it appears that the inertia of bodies, though indeed influenced [beeinflußt] by other bodies, is not caused [verursacht] by them”
- A criticism of “the theory familiar to us today”: “in the latter the inertial system is introduced, whose state of motion is, on the one hand, not conditioned [bedingt] by the states of observable objects, and therefore is caused [verursacht] by nothing accessible to perception, but on the other hand, should be determinative [bestimmend sein soll] of the behaviour of material points”[†]
- Praise for the Einstein–Grossman theory: “if the inertia of a body can increase due to cumulation of masses in its vicinity, then we have no choice but to view the inertia of a point as being conditioned [bedingt] by the existence of the other masses. Thus, inertia appears to be conditioned [bedingt] by a sort of interaction between the mass point to be accelerated and all of the other mass points”[‡]

[†] Translation of Ryckman, *Einstein*, 229
[†] CPAE translation emended

The Maxwell–Lorentz Equations





$$
\begin{aligned}
\partial_t B + \nabla \times E &= 0 \\
\partial_t E - \nabla \times B &= -J \\
\nabla \cdot E &= 4\pi\rho \\
\nabla \cdot B &= 0 \\
\dot{p} &= q(E + v \times B)
\end{aligned}
$$

Here E and B are the electric and magnetic fields;
ρ and J are the charge density and current;
v, p, and q are the velocity, momentum, and charge of a charged particle

The Electromagnetic Mach–Einstein Principle

E-M M-E Principle: The disposition (over time) of (discrete) charged matter determines the electromagnetic field

Warning: Einstein was no fan of the E-M M-E[†]

[†] Einstein entertained/favoured an approach on which charged particles were stable structures in a non-linear electromagnetic field. See his letter to Lorentz of 23 May 1909 (5.163); On the General Theory of Relativity (Addendum) of November 1915 (6.22); The Foundations of the General Theory of Relativity of 1916 (6.30, 188); the letter to Kaluza of 29 May 1919 (9.48); his tribute to Lorentz of 1920 (Ether and Relativity, 7.38, 23).

Does the EM-ME Principle Hold?

No. In the vacuum case the equations are:

$$
\begin{aligned}
\partial_t B + \nabla \times E &= 0 \qquad & \nabla \cdot E &= 0 \\
\partial_t E - \nabla \times B &= 0 \qquad & \nabla \cdot B &= 0
\end{aligned}
$$

These admit a vast a family of solutions. But note that we lose nothing, in terms of applicability, if we declare all vacuum solutions to be unphysical (or all but one)

What if we add matter?

Hard to say. We need to do something about the problem of self-interaction of charged particles. And we need to decide what the EM-ME says—and whether we are hoping to derive it from the Maxwell–Lorentz equations alone (honest toil) or whether we are allowed to supplement the equations by further postulates (theft-ish)

Lorentz's Approach

* Work with point particles, ignore self-interactions, and restrict attention to solutions of Maxwell's equations that arise via the method of retarded potentials:

  > *We need not ... speak of other solutions, if we assume that an electromagnetic field in the ether is never produced by any other causes than the presence and motion of electrons* [†]

* The result is a theory in which a strong causal form of the Electromagnetic Mach–Einstein Principle obtains: the distribution of charged matter causally determines the electromagnetic field—because we have added as a postulate that fields are caused by charges

[†] *The Theory of Electrons*, §14

Retarded Potentials?

* First, select a fundamental solution for Maxwell's equations—a solution for a point charge that exists only at one instant
  * There are many such solutions.
  * But if charge causally determines the field, then we want the unique one that lives on the future lightcone and is invariant under symmetries of the charge configuration
* Now we can find the field corresponding to the worldline of a charged particle: treat the worldline as a superposition of instantaneously existing point charges, and sum the corresponding fundamental solutions

An Approach with Honest Interactions

* We can follow Abraham by working with extended particles that are rigid spheres in a preferred frame
* Now we get a well-behaved theory that includes self-interaction
* Supervenience still fails radically in the vacuum case
* But (pausibly): the only field consistent with a single charge permanently at rest is the charge's Coulomb field F_C:
  i) If the exterior field isn't spherically symmetric, then presumably the charge will accelerate at some point
  ii) If the exterior field is spherically symmetric, if must be F_C (by the electromagnetic analog of Birkhoff's theorem)
* If this works, then we have managed to prove that the EM-ME Principle holds in one simple case. If it doesn't work, it will be because there is some surprising way of evading (i)—and we can handle that by resorting to a little theft by adding a postulate that fields must share the symmetries of their sources