Assessing the Hole Argument

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Introduction

The hole argument urges that taking spacetime points as objects implies radical indeterminism. The most common reply to it is that points are individuated by their pattern of properties and relations, and this blocks the indeterminism. This reply is made precise by the **drag-along response**: only given an isomorphism f sending a point p in one model to a point q in another model, $f(p) = q$, can p and q "correspond". More vividly: we should take q as "really being" p; or "replace q with p".

Weatherall (2018) argues the drag-along response is mathematically mandatory. We reject this, although we agree that mathematics is indifferent to the identity of its objects, including the points in models of spacetime theories—so that the response is natural.

But the response is also limited. General relativity, and indeed other spacetime theories, uses other means of "trans-world identification" of points: which we call **threading**—and illustrate with the Lie derivative, Noether's theorems, and limits of spacetimes.

Besides: symmetric models make trouble for the drag-along response (in a simple version).

They also provide a *warning* to the substantivalist not to interpret "too haecceitistically" two regions whose points are related by threading.

Outline

The hole argument's threat of indeterminism is easily visualised, as in Figure 1. We take an isomorphism between two models that have the same manifold (and so the same base-set of spacetime points). Understood "naively", the two models represent two different distributions of spatiotemporal relations, as encoded by the metric (and different distributions of material properties and relations, as encoded by matter fields, if the models include such)—over one and the same base spacetime manifold.

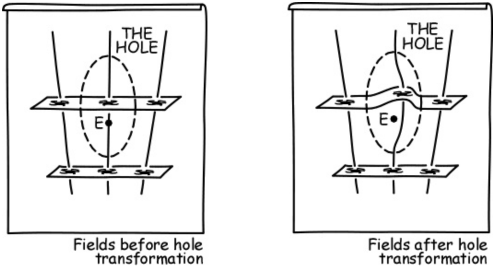

Figure: The metrics representing the spacetimes on the left and on the right are isomorphic. Does the Galaxy pass through spacetime point E? (Thanks to John Norton)

But now enters the— philosophical, and so far, vague—idea: *points are individuated by their pattern of properties and relations, as encoded in the metric and matter fields*.

This idea prompts the proposal:— Only once we are given an isomorphism (not merely a diffeomorphism!) f sending a point p in one model to a point q in another model, $f(p) = q$, do the points p and q "correspond". And then we should take q as "really being" p, or "replace q with p": so that the two models represent the same physical possibility.

We call this proposal **the drag-along response** to the hole argument. (A way to endorse 'anti-haecceitism', 'sophisticated substantivalism'; and what Pooley (2020) calls 'denying *Plurality*', i.e. denying that the two models represent different possibilities.)

We will argue that this response is limited. We need other means of "trans-world identification" of points.

The limitations of drag-along

Indifference about identity does not make drag-along mandatory!

Modern mathematics is indifferent to the identity of its objects: (recall Hilbert vanquishing Frege!). But that does *not* mean that the mathematics, with its indifference, makes it *mandatory* to use drag-along for the "trans-world identification" of points.

And similarly for a less "modally involved" notion of "corresponding point", when two contexts or situations are being compared: maths does not make it mandatory to use drag-along as the definition.

Indeed, as one might guess: mathematics is flexible, even opportunistic, in taking objects as the same as (or more cautiously: as corresponding to) each other, in whatever manner it needs for the endeavour (e.g. definition or proof) in hand.

We will show shortly how differential geometry uses, and must use, means of identifying points that are different from drag-along.

As an umbrella term for these other means, we say: *threading relation*.

(i): So our use of ‘individuation’, ‘identification’ is not meant to signal fully-fledged identity. But otherwise, we use them in philosophers’ usual sense.

(ii): Thus ‘individuate’ means: ‘provide tractably weak sufficient conditions of identification (or contrapositively: tractably strong necessary conditions of non-identification)’.

Example: the claim that persons are individuated by their bodies. That means that having the same body implies being the same person: i.e. being the same object—and so by Leibniz’s law, matching on all properties whatsoever.

The point of such a claim is evidently to give a sufficient condition for identity that is tractable, in a way that the sufficient condition given by Leibniz’s law is not.

It is far easier to check that (putatively two) objects are the same body than to check that they match on all properties whatsoever.

“Find the same body, and you will have found the same person”.

Needing not to drag-along: the Lie derivative and "threading"

There are various contexts within differential geometry, whether pure or applied, in which maths' flexible treatment of identity, or correspondence, of points—its taking facts about identity "in its stride", so as to get on with the mathematics—is very clear.

But such treatments are not haeccetist. Nor are they committed to the philosophical (Kripkean) claim that "trans-world identity" can be stipulated. Rather, they reflect mathematics' being *indifferent to identity*.

Example: To define the **Lie derivative** of a field, one drags the field along the integral curves of the vector field with respect to which one is taking the Lie derivative. This means appealing to some assumed "identity" ("correspondence") of the points: so that 'sliding the field along a curve through the points' makes sense.

Thus Arnold (1989, p. 198) says that the Lie derivative is sometimes called 'the fisherman's derivative', because one can think of the integral curves of the vector field as the flow lines of a fluid, e.g. a river.

> *The flow carries all possible differential geometric objects past the fisherman, and the fisherman sits there and differentiates them.*

The stationary earth of the river-bed, is the analogue of the "fixed" points over which the fields (Arnold's 'differential geometric objects') slide.

But according to the drag-along response: the field is *not* dragged with respect to the points of the manifold.

More precisely: for f_t the flow of a vector field X^a, we obtain:

$$\mathcal{L}_X g(x) = \lim_{t \to 0} \frac{1}{t}(g(x) - f_t^* g(f_t^{-1}(x)) \equiv 0 \; ! \tag{1}$$

NB: There is no uniform meaning to this assumed "identity" of points that the Lie derivative presupposes—and no need for such a meaning.

What matters is that differential geometry endemically invokes a scheme of identification *other than drag-along*.

As a label for such a scheme, we say: **threading scheme**.

So two points (one in domain, one in range, of a diffeomorphism) are related by **threading**; they **thread** each other.

Note the indefinite article. By saying **a** threading scheme/relation, we signal that there is no single scheme.

To do differential geometry, pure or applied, one must accommodate non-vanishing Lie derivatives.

Example: Recall the definition in relativity theory of a rigid continuous body, as a body for which the 4-velocity field X of its material particles Lie-drags the metric: $\mathcal{L}_X g = 0$.

Therefore, to describe the motion of a *non*-rigid continuous body, we need: $\mathcal{L}_X g \neq 0$.

Thus with only the drag-along, one cannot pick out the isometries of g.

There are much more sophisticated examples ...

Noether's second theorem in general relativity ...

... gives a central example of our theories needing a non-zero Lie derivative. The action is

$$\int \delta^4 x \, (\mathcal{L}_g + \mathcal{L}_m) \quad \text{with} \;\; \mathcal{L}_g = R\sqrt{g}. \tag{2}$$

We vary along a vector field X that is the generator of a flow diffeomorphism: a symmetry.

If X has compact support, the boundary terms vanish and we have

$$0 = \delta S = \int \delta^4 x((\frac{\delta \mathcal{L}_g}{\delta g} + \frac{\delta \mathcal{L}_m}{\delta g})\mathcal{L}_X g) \equiv \int \delta^4 x((G^{ab} + T^{ab})\mathcal{L}_X g_{ab}). \tag{3}$$

With the drag-along response, i.e. the fields *not* being dragged over the spacetime points, $\mathcal{L}_X g \equiv 0$. So (3) would be satisfied identically, and yielding no further conditions on the quantities involved.

But with threading, we get the usual formula: $\mathcal{L}_X g_{ab} = \nabla_{(a} X_{b)}$ (with ∇ the Levi-Civita covariant derivative). Applied to (3), we get:

$$\nabla_a G^{ab} = \nabla_a T^{ab}. \tag{4}$$

Then, using the Bianchi identity $\nabla_a G^{ab} = 0$, we also get: $\nabla_a T^{ab} = 0$, i.e. the local conservation law for the energy-momentum tensor.

We do **not** claim: the drag-along response implies that energy is not locally conserved.

But: it implies that a proof of conservation cannot go via Noether's second theorem.

Limits of spacetimes

This topic gives another sophisticated example of threading, which makes vivid its difference from dragging along.

For there is **no** need for an isomorphism, or even an isometry ...

Geroch (CMP, 1969) defines the limit of a 1-parameter family of spacetimes (in general, *not* isometric to each other), (M_λ, g_λ) with $\lambda > 0$, as the boundary of a certain 5-dimensional manifold $\mathcal{M}$, of which the M_λ are leaves.

So the limit spacetime is $\partial\mathcal{M}$ and corresponds to $\lambda = 0$.

To write down such limits, Geroch uses curves that cut the leaves (the spacetimes), thus *threading together* points in non-isometric spacetimes. Thus $p \in M_\lambda$ gets threaded to $p' \in M'_\lambda$.

(Cf. also Curiel (BJPS 2018, Section 3).)

Delimiting the scope of drag-along

The main story so far, by our lights:—

All hands agree that "GR doesn't care" about which points instantiate which properties, about the differences in trans-world identities between points of its models' manifolds. But: this does not settle interpretative, metaphysical issues. Thus Pooley (2020) on Weatherall (2018) and Fletcher (2018):

'Both authors argue that, if there are pluralities of merely haecceitistically distinct possibilities, the mathematical formalism of GR, correctly interpreted, is necessarily indifferent to differences between them. But this just means that GR does not distinguish between any two elements of such a plurality; both will count as physically possible according to GR or neither will. And that, of course, is just to admit that, according to any metaphysical view committed to such pluralities, GR is indeterministic. The indeterminism cannot be avoided by remaining loftily above the metaphysical affray.'

(For similar comments, cf. Arledge & Rynasiewicz 2018, Sections 3 and 4.)

When precisely is drag-along the correct, or at least an appropriate, scheme for "identifying" points?

We have no general answer. Maybe there is no general answer—even once one assumes certain views about the metaphysics of modality and about our theories' degree of modal involvement.

Evidently, there is much to do in exploring what the general answer may be.

But we note that:

(1) symmetric models make trouble for the drag-along response, in a simple version;

(2) the substantivalist should not interpret "too haecceitistically" two regions whose points are related by threading.

1: Symmetric models threaten an 'abysmal embarrassment'
Suppose there are two (or more) isomorphisms f_1, f_2 from a model M_1 to a model M_2: with f_i mapping point p, say, in (the manifold of) M_1 to q_i in M_2, where $q_1 \neq q_2$.

Then the model M_2 has a symmetry: $f_2 \circ f_1^{-1} : M_2 \to M_2$ is an automorphism of M_2.

Here, the drag-along response faces a *contradiction*. Namely, both q_1 and q_2 are to "be" p.

This difficulty arises for any symmetric model. It was propounded as a 'abysmal embarrassment' refuting "structuralism", by Wuthrich (2008).

Muller (2011) replied that structuralism escapes by allowing points to differ *solo numero* (cf. also Saunders, Caulton & JB).

Agreed: but the point remains that the drag-along response needs to be refined ...

2: Substantivalists: beware of interpreting threaded regions "too haecceitistically"

Suppose two points in isomorphic, and so isometric, models are threaded, but *not* by the isomorphism. Say they are identical. Or similarly, two regions: call it U.

Then, even though g and g' are isometric i.e. on the whole manifolds: still, $g|_U$, and $g'|_U$ are "physically different", in that they ascribe different properties to U. For example, g ascribes it spacetime volume v, while g' ascribes it v', with $v' \neq v$.

Similarly for other fields/geometric objects.

In short: in the two models, thoroughly different functions (though both "diffeomorphism-invariant") have support on U.

So a substantivalist who wants U to be "the same region" in the two models cannot be *sophisticated*. That is: she must accept physical differences between isomorphic models.

That is: to be sophisticated, she must confine herself to "entire-manifold" statements that do not specify where and when ('in U or not?') a field-value is attained.

Conclusion

1. All hands agree that the mathematics of GR “doesn’t care” about which points instantiate which properties. But we say:—

2. This does not settle interpretative, metaphysical issues. More specifically:—

3. The drag-along response is not mathematically mandatory. And ...

4. It is also limited. General relativity, and indeed other spacetime theories, use other means of “trans-world identification” of points: as we saw with the Lie derivative, Noether’s theorems, and limits of spacetimes.

5. The contrasting threading scheme must be careful about interpreting the threaded regions: isomorphic models will have *physically* different regional properties (but no physical difference for the entire manifold, M).

Thank you!

Threading for limits of spacetimes

Geroch's definition (1969) of a limit spacetime (cf. Curiel 2018, Sec. 3) involves a *continuous infinity* of threading relations. Each one is given by a congruence of curves, and there is a continuous infinity of congruences.

Geroch begins from the fact that 'one cannot speak simply of the limit of [e.g.] the Schwarzschild solution as the mass $\lambda \to 0$', for the spacetime one obtains in the limit [e.g. Minkowski or Kasner] depends on the choices of coordinates' (p.182).

Geroch also declares that he considers:

(i) a 1-parameter family of spacetimes M_λ $(\lambda > 0)$, not

(ii) a 1-parameter family of metrics on a given manifold;

since he does not wish to presuppose a scheme that identifies a point $p_\lambda \in M_\lambda$ with a point $p_{\lambda'} \in M_{\lambda'}$ $(\lambda \neq \lambda')$.

Note that even (i) could spell trouble for the drag-along response, since the different M_λ are not required to be isometric.

The main idea of Geroch's definition is to make each M_λ a sub-manifold (a "leaf") of a 5-dimensional manifold $\mathcal{M}$; so that the limit spacetime is defined as, roughly, the boundary $\partial\mathcal{M}$ of $\mathcal{M}$. Geroch writes:—

'The 5-manifold $\mathcal{M}$ contains all the information of our original collection $(M_\lambda, g^{ab}(\lambda))$, but does not define a preferred correspondence between different M_λ. [*Footnote*: Such a correspondence could be defined by giving a vector field on $\mathcal{M}$, nowhere vanishing and nowhere tangent to the M_λ: $p_\lambda \in M_\lambda$ and $p_{\lambda'} \in M_{\lambda'}$ are in correspondence if a trajectory of this vector field joins p_λ and $p'_{\lambda'}$. However, no such vector field is in the structure of $\mathcal{M}$.]'

Thus the correspondence, the threading relation, is arbitrary or fiducial.

'The problem of finding limits of the family $(M_\lambda, g^{ab}(\lambda))$ amounts to that of placing a suitable boundary on $\mathcal{M}$. We define a *limit space* of $\mathcal{M}$ as a 5-manifold $\mathcal{M}'$ with boundary $\partial\mathcal{M}'$, equipped with a tensor field $g^{'\alpha\beta}$, a scalar field λ', and a smooth, one-to-one mapping Ψ of $\mathcal{M}$ onto the interior of $\mathcal{M}'$ such that three conditions are satisfied:

1. Ψ is an isometry, i.e. Ψ takes $g^{\alpha\beta}$ into $g^{'\alpha\beta}$, and λ into λ'.

2. $\partial\mathcal{M}'$ is the region given by $\lambda' = 0$. We require, furthermore, that $\partial\mathcal{M}'$ be connected Hausdorff and non-empty.

3. $g^{'\alpha\beta}$ has signature $(0, =, -, -, -)$ on $\partial\mathcal{M}'$.' (p. 182)

To write down limits, one needs to use a family of frames (tetrads) on the M_λ—and so a congruence of curves in $\mathcal{M}$, each of which cuts each leaf once. Thus each such congruence specifies a threading relation: $p_\lambda \in M_\lambda$ and $p_{\lambda'} \in M_{\lambda'}$ are related—correspond to each other—iff they lie on the same curve of the congruence. Geroch writes:—

'By a *family of frames* in $\mathcal{M}$ we mean an orthonormal tetrad $w(\lambda)$ of vectors tangent to M_λ and attached to a single point $p_\lambda \in M_\lambda$, for each $\lambda > 0$, such that the $w(\lambda)$ vary smoothly along the smooth curve in $\mathcal{M}$ defined by the points p_λ. [In our language, this is one curve of the congruence: one set of points that thread each other.]
Let $\mathcal{M}'$ be a limit space of $\mathcal{M}$, and let $w(\lambda)$ be a family of frames which assumes a limit—i.e., approaches a frame $w(0)$ at some point $p_0 \in \partial\mathcal{M}'$—as $\lambda \to 0$. Let us represent points in M_λ in a neighborhood of p_λ in terms of the system of normal coordinates based on $w(\lambda)$. In terms of these coordinates, the components of the metric tensor in the M_λ approach a limit as $\lambda \to 0$, and the limiting components are precisely the components of $g^{ab}(0)$ in $\partial\mathcal{M}'$ in a neighborhood of p_0. Thus, the family of frames $w(\lambda)$ uniquely defines the limit space $\mathcal{M}'$, at least in a suffiiciently small neighborhood of p_0.' (p.183)

In the light of Geroch's definitions, Curiel proposes two criteria for *the existence of spacetime points independent of metrical structure*. This phrase plays for Curiel the role of our 'a scheme of identification other than drag-along'.

Curiel calls a family M_λ of spacetimes an *ancestral family* of any of its limit spacetimes $\partial\mathcal{M}'$. And where Geroch talks of a curve p_λ cutting each leaf M_λ—and we say 'a curve of a threading congruence'—Curiel says 'a way of identifying points between two spacetimes/manifolds'.

He thinks of the limit spacetime $\partial\mathcal{M}'$ as an idealization of the various M_λ that foliate $\mathcal{M}$; and of the M_λ as de-idealizations of $\partial\mathcal{M}'$. One can of course de-idealize ('add details') in different ways: e.g. adding charge, incrementally, to Schwarzschild, or incrementally adding non-sphericity. This corresponds to different ancestral families M_λ (and so to different congruences/threading relations), of whose limit space the limit spacetime (the idealization) is the boundary. Thus adding charge and adding non-sphericity to Schwarzschild involve different 5-manifolds $\mathcal{M}$, so different limit spaces $\mathcal{M}'$; and so different boundaries $\partial\mathcal{M}'$ of the latter. Thus Schwarzschild is the limit of (at least!) two families M_λ.

So the idea of Curiel's criteria is that the points in the limit spacetime, $p_0 \in \partial\mathcal{M}'$, exist independently of metrical structure iff there is a 'canonical way to identify spacetime points during gradual modification to the local spacetime structure' (Definition 1, p. 462).

This is made more precise, and stronger, by requiring that such an identification exists for all such gradual modifications, i.e. for each M_λ of which the spacetime is the limit spacetime.

That is: $p_0 \in \partial\mathcal{M}'$ exists independently of metrical structure iff: for any M_λ of which $\mathcal{M}'$ is the limit space, there is a family of frames $w(\lambda)$ on M_λ varying smoothly on a curve $p_\lambda \in \mathrm{M}_\lambda$, with limit $w(0)$ at p_0. (Cf. Definition 2, p. 463.)